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Statistical Properties of the Noise Output Current

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SUMMARY

The response of an electronic system to the photons emitted by a radioactive source (noise source) is exponential. Some statistical properties of the system output current are investigated for Poisson-distributed decay. It is found that the power spectral density for the noise is approximately proportional to the reciprocal of the frequency square (ω^2). Due to the monotonic, decreasing nature of the noise power spectrum, it is found that the higher the signal frequency, the larger is the signal-to-noise ratio for such a system.

1. Introduction

The concept of shot noise in electronic systems is quite well known and its properties have been studied by several authors ([1–3,6]). In this paper a similar concept for the radioactive decay response to an electronic system is considered and, under certain assumptions, some of the statistical properties of the system output current are investigated.

We consider a radioactive source (noise source) with the following amplitude function:

$$a(t) = \frac{C}{2}(1 + \alpha \sin \omega_0 t), \tag{1}$$

where C is the peak-to-peak amplitude of the noise, α is a constant and ω_0 is the frequency.

We assume that the system response to a photon emitted from the noise source is the following exponential function:

$$i(t) = I_0 \exp(-t/T), \tag{2}$$

where

$i(t)$ = output current of the system caused by the photon t seconds after its arrival,
I_0 = constant,
T = time constant of the decay.

We further assume that the radioactive decay is subject to Poisson distribution. In the real world such an electronic system can be applied to a semiconductor device, for example. In a semiconductor device, if the radioactive source has sufficient energy, then ionization takes place in the device so that hole-electron pairs will be generated. After the removal of radiation it is found, experimentally, that the carrier concentration decays exponentially with a constant lifetime T. The lifetime T is composed of the surface recombination time T_s and the volume recombination time T_v. These are well known as generation and recombination process ([5]).

It is of utmost importance to obtain the maximum signal-to-noise ratio in the electronic system. Consequently, to understand the behavior of the noise and to describe it mathematically is the essential step for optimizing the performance of the electronic system. It is evident that the signal-to-noise ratio can be derived from their respective power density spectra. However the power spectral density of the noise at the output of an electronic system may be determined from the output auto-correlation function. In this paper we investigate the auto-correlation function, the power density spectrum, and the characteristic function of the output current of the system.

2. System Output Current

Let

$$i(t, t_k) = I_0 \exp[-(t-t_k)/T] \tag{3}$$

be the magnitude of the current generated in the system due to a photon, when exposed to a unit amplitude, and t_k be the time of emission of a photon. Then the system output current $I(t)$ at time t is given by

$$I(t) = \sum_k a(t_k)\, i(t, t_k), \tag{4}$$

where $a(t)$ is given by Eq. (1).

From Eqs. (1) and (4) it follows that

$$I(t) = \sum_k (CI_0/2)(1+\alpha \sin \omega_0 t_k) \exp[-(t-t_k)/T]. \tag{5}$$

There are an infinite number of terms in the series (5). From a computational point of view, it is easier to consider a corresponding finite sum. We thus concentrate upon a large but finite interval $(-T^* < t < T^*)$, and define an event A_n such that exactly n photons are emitted in this time interval, i.e.,

$$-T^* < t_k < T^*, \quad \text{for} \quad k = 1, 2, \ldots, n. \tag{6}$$

We further assume that the emission outside the above interval is negligible. In fact, what we are stating is that all t_k are independently and uniformly distributed over the interval $(-T^*, T^*)$.

Now (5) can be written as:

$$I(t) = \sum_{k=1}^{n} f(t, t_k), \tag{7}$$

where

$$f(t, t_k) = (CI_0/2)(1+\alpha \sin \omega_0 t_k) \exp[-(t-t_k)/T]. \tag{8}$$

We now discuss the statistical properties of the output current $I(t)$ given in Eq. (7). In that respect, we first compute the (auto) correlation function, which is a rough measure of the dependence of values of the output separated by a fixed time interval.

3. The Correlation Function of I(t):

The correlation function of $I(t)$ is given by

$$\begin{aligned}\Phi_{II}(\tau_1, \tau_2) = E[I(\tau_1)I(\tau_2)] &= \sum_{n=0}^{\infty} P(A_n) E[I(\tau_1)I(\tau_2)|A_n] \\ &= \sum_{n=0}^{\infty} P(A_n) \sum_{r=1}^{n} \sum_{s=1}^{n} E[f(\tau_1, t_r) f(\tau_2, t_s)|A_n]. \end{aligned} \tag{9}$$

We know that the radioactive decay is poisson-distributed. If the mean rate of emission of the photons is λ, then

$$P(A_n) = \frac{(2\lambda T^*)^n}{n!} \exp(-2\lambda T^*). \tag{10}$$

Now from (9) we note that in the double sum

$$\sum_{r=1}^{n} \sum_{s=1}^{n} E[f(\tau_1, t_r) f(\tau_2, t_s)|A_n] \tag{11}$$

the terms for which r and s are different, the random variables $f(\tau_1, t_r)$ and $f(\tau_2, t_s)$ are independent. Therefore,

$$E[f(\tau_1, t_r)f(\tau_2, t_s)|A_n] = E[f(\tau_1, t_r)|A_n]E[f(\tau_2, t_r)|A_n]$$
$$= \frac{1}{(2T^*)^2}\int_{-T^*}^{T^*}\int_{-T^*}^{T^*} f(\tau_1, t_r)f(\tau_2, t_s)dt_r dt_s, \quad r \neq s$$
$$= (CI_0/4T^*)^2 \exp[-(\tau_1+\tau_2)/T]$$
$$\times \int_{-T^*}^{T^*}\int_{T^*}^{T^*}(1+\alpha \sin\omega_0 t_r((1+\alpha\sin\omega_0 t_s)\exp[(t_r+t_s)/T]dt_r dt_s\,, \tag{12}$$

which equals A, say.

However, for $r=s$ we have

$$E[f(\tau_1, t_r)f(\tau_2, t_s)|A_n] = (\tfrac{1}{2}T^*)(CI_0/2)^2\exp[-(\tau_1+\tau_2)/T]$$
$$\times \int_{-T^*}^{T^*}(1+\alpha\sin\omega_0\tau)^2\exp(2\tau/T)d\tau\,, \tag{13}$$

which equals B, say.

Since in the double sum (11) there are (n^2-n) terms of the type A and n terms of the type B, so we have from (9),

$$\Phi_{\mathrm{II}}(\tau_1, \tau_2) = \sum_{n=0}^{\infty} P(A_n)[(n^2-n)A+nB]\,. \tag{14}$$

Now from (10) and (14) we get

$$\Phi_{\mathrm{II}}(\tau_1, \tau_2) = \sum_{n=0}^{\infty}\frac{(2\lambda T^*)^n}{n!}\exp(-2\lambda T^*)[(n^2-n)A+nB] = (2\lambda T^*)^2 A+(2\lambda T^*)B\,. \tag{15}$$

From (12) it follows that

$$A = \left(\frac{CI_0}{4T^*}\right)^2\exp(-(\tau_1+\tau_2)/T)\int_{-\infty}^{\tau_1}\int_{-\infty}^{\tau_2}(1+\alpha\sin\omega_0 t_r)(1+\alpha\sin\omega_0 t_s)$$
$$\cdot\exp[t_r+t_s)/T]dt_r dt_s$$
$$= \left(\frac{CI_0}{4T^*}\right)^2\exp(-(\tau_1+\tau_2)/T)\prod_{j=1}^{2}\int_{-\infty}^{\tau_j}(1+\alpha\sin\omega_0 t_j)\exp(\tau_j/T)d\tau_j$$
$$= \left(\frac{CI_0}{4T^*}\right)^2\exp(-(\tau_1+\tau_2)/T)\prod_{j=1}^{2}T\exp(\tau_j/T)\left[1+\alpha\frac{\sin\omega_0\tau_j-\omega_0 T\cos\omega_0\tau_j}{1+\omega_0^2T^2}\right]$$
$$= \left(\frac{CI_0 T}{4T^*}\right)^2\prod_{j=1}^{2}\left[1+\alpha\frac{\sin\omega_0\tau_j-\omega_0 T\cos\omega_0\tau_j}{1+\omega_0^2T^2}\right]. \tag{16}$$

From (13) it can be seen that

$$B = \left(\frac{CI_0}{2}\right)^2\frac{1}{2T^*}\exp(-(\tau_1+\tau_2)/T)\int_{-\infty}^{m}(1+\alpha\sin\omega_0\tau)^2\exp(2\tau/T)d\tau\,,$$

where $m=\min(\tau_1, \tau_2)$. The algebraic simplification of this expression gives:

$$B = \left(\frac{CI_0}{2}\right)^2\frac{T}{2T^*}\exp(-|\tau_1-\tau_2|/T)\left[\tfrac{1}{2}+2\alpha\left(\frac{2\sin\omega_0 m-\omega_0 T\cos\omega_0 m}{4+\omega_0^2T^2}\right)\right.$$
$$\left.+\alpha^2\frac{\omega_0^2T^2+2\sin^2\omega_0 m-\omega_0 T\sin 2\omega_0 m}{4(1+\omega_0^2T^2)}\right]. \tag{17}$$

Substituting A and B from (16) and (17) in (15), we get

$$\Phi_{\mathrm{II}}(\tau_1, \tau_2) = \lambda T(CI_0/2)^2 \left[\lambda T \prod_{j=1}^{2} \left(1 + \alpha \frac{\sin \omega_0 \tau_j - \omega_0 T \cos \omega_0 \tau_j}{1 + \omega_0^2 T^2}\right) + \exp(-|\tau_1 - \tau_2|/T \left\{\frac{1}{2} + 2\alpha \left(\frac{2 \sin \omega_0 m - \omega_0 T \cos \omega_0 m}{4 + \omega_0^2 T^2}\right) + \alpha^2 \frac{\omega_0^2 T^2 + 2 \sin^2 \omega_0 m - \omega_0 T \sin 2\omega_0 m}{4(1 + \omega_0^2 T^2)}\right\}\right] \tag{18}$$

4. The Power Density Spectrum of $I(t)$

We now find the spectral density of the output current $I(t)$ by taking the Fourier transform of the auto-correlation function. The spectral density of $I(t)$ is,

$$S_{\mathrm{II}}(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-i\omega\tau} \Phi_{\mathrm{II}}(\tau) d\tau \,, \tag{19}$$

$$\Phi_{\mathrm{II}}(\tau) = \lim_{T^* \to \infty} \frac{1}{2T^*} \int_{-T^*}^{T^*} \Phi_{\mathrm{II}}(t, t+\tau) dt \,. \tag{20}$$

From (18) we get, after substituting $\tau_1 = t$ and $\tau_2 = t + \tau$,

$$\Phi_{\mathrm{II}}(t, t+\tau) = (\lambda TCI_0/2)^2 \left[1 + \frac{\alpha}{1 + \omega_0^2 T^2} \left\{\sin \omega_0 t + \sin \omega_0 (t+\tau) - \omega_0 T(\cos \omega_0 t + \cos \omega_0 (t+\tau)) \right. \right.$$
$$+ \left(\frac{\alpha}{1 + \omega_0^2 T^2}\right)^2 \{(\sin \omega_0 t - \omega_0 T \cos \omega_0 t)(\sin \omega_0 (t+\tau) - \omega_0 T \cos \omega_0 (t+\tau)\}$$
$$+ \frac{\exp(-|\tau|/T)}{2\lambda T} \left\{1 + 4\alpha(2 \sin \omega_0 m - \omega_0 T \cos \omega_0 m)/(4 + \omega_0^2 T^2) \right.$$
$$\left.\left. + \alpha^2 \left(\frac{\omega_0^2 T^2 + 2 \sin^2 \omega_0 m - \omega_0 T \sin 2\omega_0 m}{2(1 + \omega_0^2 T^2)}\right)\right\}\right] \,, \tag{21}$$

where $m = \min(t, t+\tau)$. From (20) and (21) it follows that

$$\Phi_{\mathrm{II}}(\tau) = (\lambda TCI_0/2)^2 \left[1 + B_1 \left(\frac{\alpha}{1 + \omega_0^2 T^2}\right) + B_2 \left(\frac{\alpha}{1 + \omega_0^2 T^2}\right)^2 + \frac{\exp(-|\tau|/T}{2\lambda T} \left\{1 + B_3 \left(\frac{4\alpha}{4 + \omega_0^2 T^2}\right) + B_4 \left(\frac{\alpha^2}{2(1 + \omega_0^2 T^2)}\right)\right\}\right] \,, \tag{22}$$

where

$$B_1 = \lim_{T^* \to \infty} \frac{1}{2T^*} \int_{-T^*}^{T^*} [\sin \omega_0 T + \sin \omega_0 (t+\tau) - \omega_0 T(\cos \omega_0 t + \cos \omega_0 (t+\tau))] dt = 0 \,;$$

$$B_2 = \lim_{T^* \to \infty} \frac{1}{2T^*} \int_{-T^*}^{T^*} (\sin \omega_0 t - \omega_0 T \cos \omega_0 t)(\sin \omega_0 (t+\tau) - \omega_0 T \cos \omega_0 (t+\tau)) dt$$
$$= \lim_{T^* \to \infty} \frac{1}{2T^*} \int_{-T^*}^{T^*} [(1 + \omega_0^2 T^2) \cos \omega_0 \tau - (1 - \omega_0^2 T^2) \cos \omega_0 (2t+\tau) - 2\omega_0 T \sin \omega_0 (2t+\tau)] \frac{dt}{2} = \tfrac{1}{2}(1 + \omega_0^2 T^2) \cos \omega_0 \tau \,;$$

$$B_3 = \lim_{T^* \to \infty} \frac{1}{2T^*} \int_{-T^*}^{T^*} (2 \sin \omega_0 m - \omega_0 T \cos \omega_0 m) dt = 0 \; ;$$

$$B_4 = \lim_{T^* \to \infty} \frac{1}{2T^*} \int_{-T^*}^{T^*} (\omega_0^2 T^2 + 2 \sin^2 \omega_0 m - \omega_0 T \sin 2\omega_0) dt = (1 + \omega_0^2 T^2) \, .$$

Now substituting these values of B_1, B_2, B_3 and B_4 in (22) we get

$$\Phi_{\text{II}}(\tau) = (\lambda T C I_0/2)^2 \left[1 + \frac{\alpha \cos \omega_0 \tau}{2(1+\omega_0^2 T^2)} + \left(1 + \frac{\alpha^2}{2}\right) \frac{\exp(-|\tau|/T)}{2\lambda T} \right] . \qquad (23)$$

From (19) and (23) we have

$$S_{\text{II}}(\omega) = \frac{K}{\pi} \left[\int_{-\infty}^{\infty} \exp(-i\omega\tau) d\tau + \frac{\alpha}{2(1+\omega_0^2 T^2)} \int_{-\infty}^{\infty} \cos \omega_0 \tau \exp(-i\omega\tau) d\tau + \frac{\alpha^2+2}{4\lambda T} \int_{-\infty}^{\infty} \exp(-|\tau|/T) \exp(-i\omega\tau) d\tau \right] , \qquad (24)$$

where

$$K = (\lambda T C I_0/2)^2$$

After integration, this expression reduces to

$$S_{\text{II}}(\omega) = \frac{K}{\pi} \left[2\pi\delta(\omega) + \frac{\pi\alpha}{2(1+\omega_0^2 T^2)} (\delta(\omega-\omega_0) + \delta(\omega+\omega_0)) + \left(\frac{\alpha^2+2}{4\lambda T}\right) \frac{T}{1+\omega^2 T^2} \right]$$

$$= K \left[2\delta(\omega) + \alpha \left(\frac{\delta(\omega-\omega_0) + \delta(\omega+\omega_0)}{2(1+\omega_0^2 T^2)} \right) + \frac{\alpha^2+2}{2\pi\lambda} \left(\frac{1}{1+\omega^2 T^2} \right) \right] , \qquad (25)$$

where $\delta(\;)$ is the delta function.

From (25) it is clear that the power density spectrum $s_{\text{II}}(\omega)$ of the noise output current $I(t)$ is approximately inversely proportional to the frequency ω^2. For four different values of α the spectral density $S_{\text{II}}(\omega)$ is plotted as a function of ω in Fig. 1. It is interesting to note that the

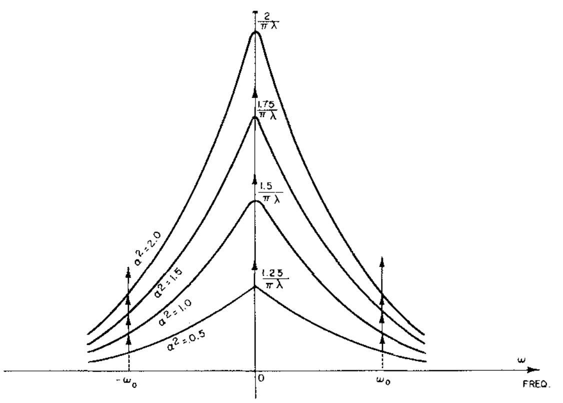

Figure 1. Power density spectrum for noise. Vertical axis: $S_{\text{II}}(\omega)$.

power density spectrum for the noise is monotonically decreasing. Furthermore, it is evident that the power density at any frequency ω has the second-order effect of the value of α as compared with the first-order effect for the noise amplitude function,

$$a(t) = \frac{C}{2}(1 + \alpha \sin \omega_0 t) \, .$$

It is to be noted that the three delta functions in the power density spectrum for fixed α are due to the constant and the sine term alone.

If the frequency ω_0 of the noise amplitude function is zero or the noise amplitude is constant, then Eq. (25) takes the following form

$$S_{II}(\omega) = K\left[(\alpha+2)\,\delta(\omega) + \left(\frac{\alpha^2+2}{2\pi\lambda}\right)\frac{1}{1+\omega^2 T^2}\right].$$

5. The Characteristic Function of I(t):

The characteristic function $\psi(u)$ of the output current $I(t)$ is,

$$\psi(u) = E[\exp(iuI(t))] = \sum_{n=0}^{\infty} PA_n)E[\exp(iuI(t))|A_n]$$
$$= \sum_{n=0}^{\infty} P(A_n)\,E\left[\exp\left(iu\sum_{k=0}^{n} f(t,t_k)\right)|A_n\right]$$
$$= \sum_{n=0}^{\infty} P(A_n)\{E[\exp(iuf(t,t_k))|A_n]\}^n. \tag{26}$$

We have already hypothesized that the random variables t_k are independently and uniformly distributed over the interval $(-T^*, T^*)$.

Therefore,

$$E[\exp(iuf(t,t_k))|A_n] = \frac{1}{2T^*}\int_{-T^*}^{T^*}\exp(iuf(t,t_k))dt_k$$
$$= 1 + \frac{1}{2T^*}\int_{-T^*}^{T^*}[\exp(iuf(t,t_k))-1]\,dt_k = 1+\beta, \tag{27}$$

where

$$\beta = \frac{1}{2T^*}\int[\exp(iuf(t,\tau))-1]\,d\tau = \frac{1}{2T^*}\int\sum_{n=1}^{\infty}\left(\frac{iuf(t,\tau)}{n!}\right)^n d\tau$$
$$= \frac{1}{2T^*}\sum_{n=1}^{\infty}\frac{(iu)^n}{n!}\int(f(t,\tau))^n d\tau. \tag{28}$$

Now combining (26) and (27), we have

$$\psi(u) = \sum_{n=0}^{\infty} P(A_n)(1+\beta)^n$$

Substituting $P(A_n)$ from Eq. (10) in this expression, we get

$$\psi(u) = \sum_{n=0}^{\infty}\frac{\{2\lambda T^*(1+\beta)\}^n}{n!}\exp(-2\lambda T^*) = \exp(2\lambda\beta T^*). \tag{29}$$

From (28) and (29), we have

$$\psi(u) = \exp\left[\lambda\sum_{n=1}^{\infty}\frac{(iu)^n}{n!}\int(f(t,\tau))^n d\tau\right]$$
$$= \exp\left[\lambda\sum_{n=1}^{\infty}\frac{(iuCI_0/2)^n}{n!}\{\exp(-nt/T)\}\int(1+\alpha\sin\omega\tau)^n\exp(n\tau/T)d\tau\right].$$

Now from this expression the expected output current and other higher moments of $I(t)$ can be computed using standard techniques.

6. Summary and Conclusions

In this paper we have studied the statistical properties of the output current for an electronic system having an exponential response from a radioactive source (noise source). The radioactive decay was assumed to be Poisson-distributed. It was found that the power spectral density for the noise is inversely proportional to the square of the frequency (ω^2). The higher the signal frequency, the larger is the signal to noise ratio for this system.

If, instead of (1), the noise amplitude function $a(t)$ has a different form, then one can obtain the power spectral density and the characteristic function by methods similar to those employed in this report. If the power spectral density is not a monotonic decreasing function of the frequency ω, then the maximum signal-to-noise ratio can be obtained by the standard approach ([4]).

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